Math 564: Advance Analysis 1
Lecture 19

Proof of Lebesgue-Radon-Nikodym thm (continued). Suppose μ, ν are finite measures. For a non-negative $\mathcal{B}$-measurable $f : X \to [0,\infty)$, recall that ν_f denotes the measure $\nu_f(B) := \int_B f\, d\nu$ $\forall B \in \mathcal{B}$. It is common to write $d\nu_f := f\, d\nu$ to mean this. By the observation before this proof, it is enough to show that $\mu = \nu_f + \mu_0$ for some $\mathcal{B}$-meas. non-neg. f and $\mu_0 \perp \nu$. Let

$$\mathcal{F} := \{ f : X \to [0,\infty) : f \ \mathcal{B}\text{-meas. and } \overbrace{\mu \geq \nu_f}^{d\mu \geq f\, d\nu} \}.$$

Note that $0 \in \mathcal{F}$ and $\mathcal{F}$ is closed under max operation. Indeed, if $f, g \in \mathcal{F}$ then letting $X = X_f \sqcup X_g$, where $X_f := \{x \in X : f(x) \geq g(x)\}$ and $X_g := \{x \in X : f(x) < g(x)\}$, then $d\mu \geq \mathbb{1}_{X_f} \cdot f\, d\nu = \mathbb{1}_{X_f} \cdot \max(f,g) \cdot d\nu$ and $d\mu \geq \mathbb{1}_{X_g} \cdot g \cdot d\nu = \mathbb{1}_{X_g} \cdot \max(f,g) \cdot d\nu$, so $d\mu \geq \max(g,f)\, d\nu$.

Let $(f_n) \subseteq \mathcal{F}$ s.t. $\lim_n \int f_n\, d\nu = \sup\{\int f\, d\nu : f \in \mathcal{F}\} \leq \mu(X)$.
By replacing each f_n with $\max(f_0, f_1, \dots, f_n)$ we may assume that (f_n) is increasing so $f := \lim_n f_n$ exists and by the monotone convergence thm:
$\int f\, d\nu = \lim_n \int f_n\, d\nu = \sup\{\int f\, d\nu : f \in \mathcal{F}\}$.

Note that $f \in \mathcal{F}$ because $\forall B \in \mathcal{B}$, we have

$$\mu(B) \geq \int_B f_n\, d\nu \text{ so } \mu(B) \geq \lim_n \int_B f_n\, d\nu = \int_B f\, d\nu.$$

We show that f is as desired. Let $\mu_0 := \mu - \nu_f$, so $\mu_0 \geq 0$ is a measure. Applying the previous lemma to μ_0 and ν, we get that either $\mu_0 \perp \nu$, in which case $\mu = \nu_f + \mu_0$ is as desired, or $\exists A \in \mathcal{B}$ with $\nu(A) > 0$ s.t. $d\mu_0|_A \geq \varepsilon \cdot d\nu|_A$. If the latter, then $f + \varepsilon \mathbb{1}_A \in \mathcal{F}$, but $\int f + \varepsilon \mathbb{1}_A\, d\nu = \int f\, d\nu + \varepsilon \cdot \nu(A) > \int f\, d\nu$, contradicting the maximality of $\int f\, d\nu$.

For σ-finite μ and ν, we can write $X = \bigsqcup_n X^{(n)}$ s.t. both μ and ν are finite on each X_n (let $X = \bigsqcup_k Y_k$ be for μ and $X = \bigsqcup_\ell Z_\ell$ be for ν, take $\bigsqcup_{k,\ell} Y_k \cap Z_\ell$). Then for each n, we get $X^{(n)} = X_0^{(n)} \sqcup X_1^{(n)}$ and a function f_n as above for $\mu|_{X^{(n)}}$ and $\nu|_{X^{(n)}}$, and $f := \sum_n f_n$ and $X_0 := \bigsqcup_n X_0^{(n)}$, $X_1 := \bigsqcup_n X_1^{(n)}$ are desired. $\square$

Cor. Let μ, ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. If $\mu \ll \nu$ then for all $g \in L^1(X, \mu)$ or $g : X \to [0, \infty]$ $\mathcal{B}$-measurable, we have

$$\int g\, d\mu = \int g \cdot \frac{d\mu}{d\nu} \cdot d\nu. \quad (*)$$

Proof. For each $B \in \mathcal{B}$, we already know $\int \mathbb{1}_B\, d\mu = \int \mathbb{1}_B \cdot \frac{d\mu}{d\nu}\, d\nu$. Linearity gives $(*)$ for simple functions, and hence for $g \geq 0$ by MCT and the fact that g is an increasing limit of simple functions. For L^1 function g, let $g = g^+ - g^-$ and apply linearity. $\square$

Chain rule. Let μ, ν, λ be σ-finite measures on a measurable space $(X, \mathcal{B})$. Suppose that $\mu \ll \nu$ and $\nu \ll \lambda$. Then $\mu \ll \lambda$ and

$$\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}.$$

Proof. The transitivity is by definition: $\lambda(B) = 0 \Rightarrow \nu(B) = 0 \Rightarrow \mu(B) = 0$. For any $B \in \mathcal{B}$, we have

$$\mu(B) = \int \mathbb{1}_B \cdot \frac{d\mu}{d\nu}\, d\nu \overset{\text{Corollary}}{=} \int \mathbb{1}_B \cdot \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}\, d\lambda = \int_B \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}\, d\lambda.$$

So $\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda} = \frac{d\mu}{d\lambda}$ a.e. by the uniqueness of the Rad-Nik derivative. $\square$

Cor. Let μ, ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. If $\mu \sim \nu$ then $\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}$.

Proof. $\mu \ll \nu \ll \mu$, so $1 = \frac{d\mu}{d\mu} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu}$. $\square$

Why call a derivative? Let μ be a Borel measure on $\mathbb{R}$ that is finite on bounded intervals. Let $F(x) := \begin{cases} \mu((0,x]) & \text{if } x \geq 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$. Suppose that $F: \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then F is Riemann integrable and for every $x \in \mathbb{R}$, the fundamental theorem of calculus holds:

$$F(x) = \int_0^x F'(t)\,dt$$

But by the HW problem, Riemann integrable functions are Lebesgue integrable and the integrals coincide, so

$$\mu((0,x]) = \int_{(0,x]} F' \, d\lambda,$$

which implies that $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = F'$. The details are left as HW. The μ is the "generalized antiderivative" of $\frac{d\mu}{d\lambda}$.

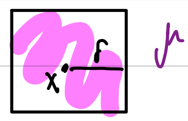
Furthermore, recall that for an invertible linear transformation $T_A : \mathbb{R}^d \to \mathbb{R}^d$, where A is the corresponding matrix, $\frac{d(T_* \lambda)}{d\lambda} \equiv |\det(A)|$. Note that A is the Jacobian of T_A. This generalizes to smooth transformations $T: \mathbb{R}^d \to \mathbb{R}^d$. Indeed, by the uniqueness of $\frac{d(T_*\lambda)}{d\lambda}$ must be equal to the $|\det(\text{Jacobian})|$.

Differentiation of measures on $\mathbb{R}^d$.

Let μ be a Borel measure on $\mathbb{R}^d$ that is finite on every bdd box and $\mu \ll \lambda$. Thus $d\mu = \frac{d\mu}{d\lambda}\, d\lambda$, where $\frac{d\mu}{d\lambda}$ is Borel and

integrable on every bdd box. It's conceivable that for a small diameter δ, if B_δ is a box of diam $< \delta$ centered at $x \in \mathbb{R}^d$ then

$$\frac{d\mu}{d\lambda}(x) \approx \frac{\mu(B_r(x))}{\lambda(B_r(x))}.$$



This turns out to be true:

Lebesgue Differentiation Theorem. For each locally integrable $f : \mathbb{R}^d \to \mathbb{R}$ (i.e. $f \cdot \mathbb{1}_B \in L^1(\mathbb{R}^d, \lambda)$ for every bounded box B), we have:

$$f(x) = \lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$$

for a.e. $x \in \mathbb{R}^d$, where $B_r(x)$ is the ball of radius r about x in d_∞-metric.

In particular, if $\mu \ll \lambda$ and μ is finite on bdd boxes, then

$$\frac{d\mu}{d\nu}(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))}.$$

This gives a method of computing the Radon–Nikodym derivative!